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New Triangular Plate-Bending Finite Element with Transverse Shear Flexibility

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Introduction

EARLY formulations of triangular plate-bending finite elements were given by Clough and Tocher[1] and by Bazeley et al.[2] Improvements to these elements have been made by using higher degree polynomials for transverse displacements; indeed elements of very high accuracy have been reported by Argyris,[3] Bell,[4] and Cowpoer et al.[5] using quintic polynomials for the displacement field. These high accuracy elements, however, have curvatures and/or higher order derivatives of displacements as grid point degrees of freedom. This situation leads to an inconvenience when step property variations are introduced. Furthermore, the elements discussed in Refs. 1–5 do not possess the property of transverse shear flexibility. Clough and Felippa[6] and Irons and Razzaque[7] have included the effects of transverse shear flexibility in finite element formulations without additional degrees of freedom to control transverse shear. The elements of Refs. 6 and 7, however, have w, $w_{,x}$, and $w_{,y}$ as grid point degrees of freedom rather than the quantities w, θ_x and θ_y required for continuity and proper treatment of boundary conditions in the presence of transverse shear.

The purpose of this Note is to describe a new triangular plate-bending finite element that has the advantages of the accuracy associated with a high-order displacement polynomial but does not have the previously discussed disadvantages and therefore is suitable for inclusion in general-purpose computer programs. The element has 18 degrees of freedom: the transverse displacement and 2 rotations at each vertex and at the midpoint of each side. Only displacements and rotations are included as grid point degrees of freedom and a quintic polynomial is used for lateral displacement. Effects of transverse shear have been taken into account in the element formulation by a procedure based on that used in NASTRAN[8] and a similar work for a quartic element communicated to the author by MacNeal.[9] The components of transverse shear strain are quadratic functions of position. Convergence to the limiting case of zero transverse shear strain is uniform.

Two problems of plate bending are analyzed using the new triangular elements, the static and free vibration analysis of a square isotropic plate with all edges 1) simply supported and 2) clamped. Results of calculations for displacements and stresses are compared with exact solutions from classical plate theory and with other available finite element solutions from the literature. Good accuracy is achieved with the new element for practical mesh subdivisions. In addition, two examples are presented which demonstrate the ability of the element to model the effects of transverse shear deformation in moderately thick to thick isotropic plates. Comparisons with 3-D elasticity solutions indicate the adequacy of the modeling of transverse shear in isotropic plates shown herein.

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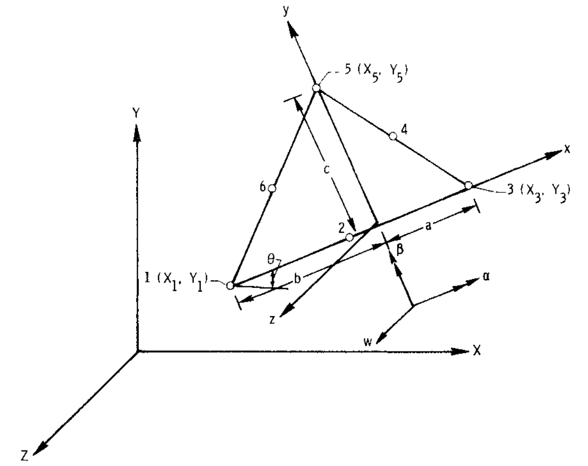

Fig. 1 Geometry of triangular element.

Derivation of Element Properties

Element geometry

A typical triangular element is shown in Fig. 1, where X, Y, and Z comprise a system of global coordinates and x, y, z comprise the system of local coordinates. The grid points of the element are numbered in counterclockwise direction as shown. The relationship between the dimensions of the triangular element a, b, c, the inclination θ between the global and local axes and the coordinates of the vertices of the element can be easily derived.[10,11]

Displacement field

The deflection $w(x, y)$ within the triangular element is assumed to vary as a quintic polynomial in the local coordinates; i.e.,

$$\begin{aligned} w(x, y) = {} & a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + \\ & a_8x^2y + a_9xy^2 + a_{10}y^3 + a_{11}x^4 + a_{12}x^3y + a_{13}x^2y^2 + \\ & a_{14}xy^3 + a_{15}y^4 + a_{16}x^5 + a_{17}x^4y + a_{18}x^3y^2 + \\ & a_{19}x^2y^3 + a_{20}xy^4 + a_{21}y^5 \end{aligned} \tag{1}$$

There are 21 independent coefficients a_1–a_{21}. These are evaluated as follows: the element has 18 degrees of freedom; viz, w, displacement in z direction; α, rotation about the x axis; and β, rotation about y axis at each of the 6 grid points. The rotations α and β are geometrically related to transverse shear strains γ_{xz} and γ_{yz} by

$$\gamma_{xz} = \partial w/\partial x + \beta \tag{2}$$

$$\gamma_{yz} = \partial w/\partial y - \alpha \tag{3}$$

It is shown in Ref. 11 that α and β can also be conveniently related to γ_{xz} and γ_{yz} through the moment equilibrium equations:

$$V_x + \partial M_x/\partial x + \partial M_{xy}/\partial y = 0 \tag{4}$$

$$V_y + \partial M_y/\partial y + \partial M_{xy}/\partial x = 0 \tag{5}$$

and, hence, can be expressed in terms of the coefficients a_1–a_{21}. Thus, 18 equations relating w, α, and β at the grid points to the 21 constants are obtained. Three additional relations are obtained by imposing the conditions that the edge rotation varies cubically along each edge.[5] Thus, the 21 coefficients can be uniquely determined in terms of the element nodal displacement vector. There is no rotation continuity between adjacent elements

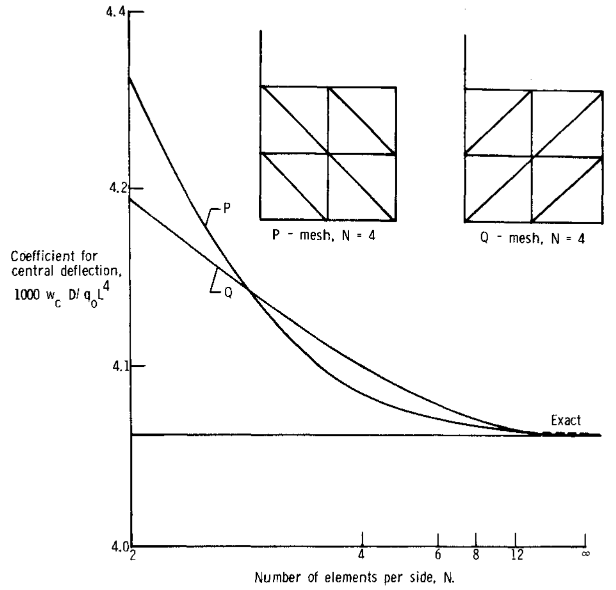

Fig. 2 Simply supported square plate: central deflection w_c under uniformly distributed load q_0.

and as a result, the element is a nonconforming element. It is necessary for convergence that any finite element be capable of representing states of uniform strain, including the special case of rigid body motions, which are states of zero strain. Both these criteria have been satisfied for the element presented herein.[11]

In addition, a square plate subjected to pure moments was analyzed with different mesh subdivisions and the finite element solution coincided with the exact solution; this establishes that an arbitrary patch of elements can sustain a state of constant strain exactly. The evaluation of stiffness matrix, consistent load vector, and consistent mass matrix follows familiar lines[12] and is not described here.

Stress recovery

After the nodal displacement vector has been obtained, the internal moments can be evaluated easily. The values of the moments at the nodal points are taken as the average values of all the element moments computed at the specific node.

Results and Discussion

Two example problems neglecting transverse shear deformation are analyzed to demonstrate the behavior of the elements, viz, the statics and free vibration of a square isotropic plate with edges 1) simply supported and 2) clamped. In addition, the effect of transverse shear in moderately thick plates is evaluated and compared with 3-D elasticity solution available in the literature.[13,14] A value of Poisson's ratio of 0.3 is used in all calculations. Due to symmetry, only $\frac{1}{4}$ plate is analyzed. Two mesh arrangements are used in the analysis: designated P (diagonal on $\frac{1}{4}$ plate passes through midedges of plate) and Q (diagonal on $\frac{1}{4}$ plate passes through center of plate). The calculated values of the deflection at the center of the plate using the present elements for the case of simply supported plate under uniformly distributed loads are shown in Fig. 2.

The high accuracy achieved with the present element for practical mesh subdivisions is evident from Fig. 2. For clamped

Table 1 a) Bending moment and corner reaction approximations for simply supported square plate ($\nu = 0.3$)

No. elements per side N	Central concentrated load P: Bending moment at mid-edge Mnn/P, Q Mesh	P Mesh	Corner reaction R/P, Q Mesh	P Mesh	Uniformly distributed load q: Bending moment at center Mxx/qL^2, Q Mesh	P Mesh	Bending moment at mid-edge Mnn/qL^2, Q Mesh	P Mesh	Corner force R/qL^2, Q Mesh	P Mesh
2	+0.0469	+0.0440	0.3555	0.0990	−0.0359	−0.0496	+0.0068	+0.0026	0.0531	0.0294
4	+0.0200	+0.0101	0.1565	0.1159	−0.0485	−0.0480	+0.0062	+0.0021	0.0737	0.0536
6	+0.0123	+0.0033	0.1360	0.1191	−0.0484	−0.0479	+0.0046	+0.0018	0.0722	0.0592
8	+0.0087	+0.0030	0.1298	0.1203	−0.0483	−0.0479	+0.0036	+0.0002	0.0704	0.0615
Analytical[10]	0.0		0.1219		−0.0479		0.0		0.06496	

Table 1 b) Bending moment approximations for clamped square plate ($\nu = 0.3$)

No. elements per side N	Concentrated load P: Bending moment at mid-edge Mnn/P, Q Mesh	P Mesh	Uniformly distributed load q: Bending moment at center Mxx/qL^2, Q Mesh	P Mesh	Bending moment at mid-edge Mnn/qL^2, Q Mesh	P Mesh
2	+0.1250	+0.0961	−0.0038	−0.0253	+0.0491	+0.0352
4	+0.1278	+0.0998	−0.0225	−0.0230	+0.0529	+0.0393
6	+0.1288	+0.1086	−0.0233	−0.0229	+0.0534	+0.0419
8	+0.1288	+0.1121	−0.0233	−0.0229	+0.0533	+0.0435
Analytical[10]	+0.1257		−0.0231		+0.0513	

Table 2 Effects of transverse shear in square isotropic plates under uniformly distributed load q ($\nu = 0.3$)

h/L	Nondimensional coefficient (Gw/hq): Simply supported plate, Thin plate theory	Present solution	3-D elasticity solution	Clamped plate, Thin plate theory	Present solution	3-D elasticity solution
0.10	170.62	178.41	178.45	52.92	60.10	62.83
0.14	44.41	48.26	48.40	13.78	17.36	18.64
0.20	...	...	...	3.31	5.08	5.604

plates, the results for the coarsest grid are not as accurate as for the simply supported plate; however, as the element size is decreased, the values of deflection obtained with the present elements approach the exact results very rapidly. Detailed discussion of results for static and free vibration analysis of square isotropic plates is given in Ref. 11. The values of calculated bending moments and corner reactions are given in Table 1. Good agreement with exact values is observed for practical mesh subdivisions.

The effect of transverse shear in moderately thick isotropic plates can be represented using the present elements. The values of the nondimensional deflection coefficients (Gw/hq) for various h/L ratios for simply supported and clamped plates are compared to the 3-D elasticity solution[13,14] in Table 2. Only $\frac{1}{4}$ plate was analyzed and Q mesh with number of elements per edge equal to 8 was used. Good agreement is seen with the 3-D elasticity solution.

Concluding Remarks

A new triangular plate-bending finite element using a quintic displacement field but having only displacement and rotations as grid point degrees of freedom is described in this Note. The examples presented demonstrate that high accuracy is achievable using this element for practical mesh subdivisions. The effect of transverse shear deformations is included in the element formulation. The present element gives satisfactory approximations for solving isotropic plate problems for cases where transverse shear effects are significant. This element is ideally suited for inclusion into general-purpose computer programs due to 1) high accuracy for practical mesh subdivisions, 2) use of only displacements and rotations as grid point degrees of freedom, and 3) inclusion of transverse shear flexibility in the element properties.

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Suppression of Ionization Instability in an MHD Disk Generator

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STABILIZATION of the ionization instability in a nonequilibrium MHD plasma by fully ionizing the seed has been observed in simulation experiments[1,2] where the current was supplied from the external circuit. Here we present the experimental results of the recovery of the effective Hall parameter in the regime of fully ionized seed in an actual nonequilibrium MHD disk generator.

The experimental conditions of the present work are summarized in Table 1. All measurements were made in the Hall open circuit as a function of the magnetic field. The effective Hall parameter was calculated from the radial electric field. The electron temperature was estimated from the intensity of the potassium resonance line (7699A).

Figure 1 shows the variation of the time-averaged line intensity against magnetic field. We can see that the intensity decreases as the magnetic field increases above 2.5 kg. This

Table 1 Experimental conditions of the present work

Working gas		Disk generator	
Ar + Potassium heated by the pressure driven shock tube		radius of the inner electrode	5 cm
		radius of the outer electrode	11 cm
Conditions in the MHD channel		channel height	
pressure	40 torr	1.97 cm at the inner electrode	
temperature	900 K	1.00 cm at the outer electrode	
velocity	1400 m/sec		
Mach number	2.4		
seed fraction	8×10^{-5}		

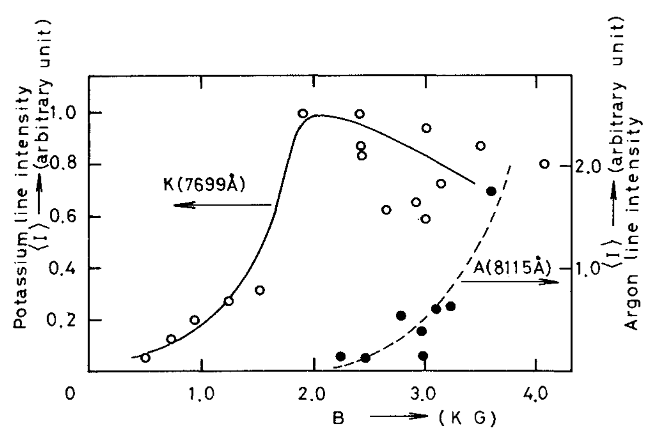

Fig. 1 Intensity of the potassium resonance line (7699Å) and the argon line (8115Å) as a function of magnetic field.

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